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Localization Estimates for a Random Discrete Wave Equation at High Frequency

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It is shown that at high frequencies matrix elements of the Green's function of a random discrete wave equation decay exponentially at long distances. This is the input to the proof of dense point spectrum with localized eigenfunctions in this frequency range. The proof uses techniques of Fröhlich and Spencer. A sequence of renormalization transformations shows that large regions where wave propagation is easily maintained become increasingly sparse as resonance is approached.

KEY WORDS: Disorder; random media; wave; localization; resonance; renormalization.

1. INTRODUCTION

Wave propagation in a random medium is important on the atomic level and on the macroscopic level. On the atomic level the waves are electron waves obeying the Schrödinger equation. It is the random impurities and irregularities in a metal that are responsible for the resistance to the flow of a current. On the macroscopic level the waves of interest are electromagnetic waves or other waves obeying the wave equation. In both cases there is a widely accepted nonrigorous theory of radiative transport. The basis of this theory is a perturbation expansion in the amount of the disorder in the medium. It predicts that the phase of the wave averages to zero, while the intensity of the wave diffuses through the medium.

Anderson[1] observed that this radiative transport theory should fail in a situation with a large amount of disorder. In fact, in this situation there should be only standing waves. This phenomenon is called *localization*. It is now believed on the basis of various theories, including refinements of the

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perturbation approach,[2] that localization is typical in one- and two-dimensional problems. This is because strong backscattering dominates, due to a low-frequency divergence. The one-dimensional case is now well understood.[3]

The most conspicuous open problem is to prove that there is diffusion in three dimensions at low disorder, as predicted by the transport theory. It would also be nice to have a better understanding of localization in dimensions greater than one. Recently Fröhlich and Spencer[4] and others have made progress on localization for discrete Schrödinger equations. A key role is played by the existence of large forbidden regions, which are due to the fact that the spectrum of the unperturbed operator is bounded and the potential has large fluctuations. Martinelli and Holden[5] have a similar result for the continuum equation; they use the fact that the spectrum of the unperturbed operator is bounded below, and so fluctuations can create large regions where low-frequency propagation is forbidden.

The wave equation should behave similarly to the Schrödinger equation, except that the medium and the frequency are related in a somewhat different way.[6] Extending the results of Fröhlich and Spencer to discrete wave equations requires the solution of certain technical problems, which are solved in the present paper.

The key results of Fröhlich and Spencer[4] are estimates on the Green's function (resolvent) of a random discrete Schrödinger equation with large disorder or at high or low frequency. The purpose of this paper is to present a similar result for a discrete wave equation with multiplicative off-diagonal random terms. The main result is a decay estimate on matrix elements of the Green's function at high frequency. This estimate is the input to a proof of dense point spectrum at high frequency.[7] Both parts of the proof, the estimate and the spectral implications, are outlined in conference proceedings.[8] The final result is that a random medium described by such an equation has only localized standing waves in this frequency range.

The general plan is to follow the lines of the Fröhlich and Spencer proof. However, there are differences between their discrete Schrödinger equation and the discrete wave equation studied here. One is that the random terms are multiplicative rather than additive. This means that locally the spectrum is scaled rather than shifted. This changes the character of the density of states bound that is used to control the near-resonance situation. Another is that the coupling between regions is random. This must be taken into account in the path expansion that is the starting point for the analysis.

The strategy of the proof is to work at fixed frequency. The first step is to obtain a decay estimate on the Green's function in a "forbidden region" where no resonance can occur. This is accomplished by the path expansion of Section 2.

The next step is to get a complementary estimate that will give at least crude control over the Green's function in the regions near resonance. This is the resonance bound of Section 3. It says that strong resonances are unlikely in moderate-sized regions.

These estimates are then used as the input to the Fröhlich–Spencer renormalization construction. This construction is based on the introduction of successively larger and more resonant regions, as indicated in Section 4. The perturbation argument of Section 5 shows that including these near-resonant regions does not destroy the decay estimate. The resonance bound then shows that these regions tend to become sparse on large distance scales. This probability argument is outlined in Sections 6 and 7.

The position space for our discrete problem is the ν-dimensional lattice $\mathbb{Z}^\nu$. A site $\mathbf{x}$ is a point in $\mathbb{Z}^\nu$. The discrete Laplace operator is defined by

$$\Delta f(x) = \sum_{|\mathbf{y}-\mathbf{x}|=1} f(\mathbf{y}) - f(\mathbf{x}) \tag{1}$$

Let c be a function from $\mathbb{Z}^\nu$ to $[0, \infty)$, the *local propagation speed.* The discrete wave equation of interest is

$$\partial^2 w/\partial t^2 = c^2 \Delta w \tag{2}$$

It is convenient to change variable by $w = cu$ to bring this to the self-adjoint form

$$\partial^2 u/\partial t^2 = c\,\Delta cu \tag{3}$$

The problem may now be given a Hilbert space formulation. Let $\mathscr{H} = l^2(\mathbb{Z}^\nu)$. Define the operator H acting in this space by

$$Hf = -c\,\Delta cf \tag{4}$$

The discrete wave equation becomes

$$\partial^2 u/\partial t^2 + Hu = 0 \tag{5}$$

The spectral properties of H will determine the nature of the solutions.

From now on we will assume that there are constants a and M such that the local propagation speed function c satisfies

$$0 < a \leqslant c^2 \leqslant M < \infty \tag{6}$$

Under this assumption the quadratic form of H is equivalent to the quadratic form of $-\Delta$ and so H is a positive self-adjoint operator.[9] Since $-\Delta$ is bounded above by 4ν, H is bounded above by $4\nu M$.

We actually want the local propagation speed function to be random. We take the simplest such model, in which the $c(\mathbf{x})$ at the sites $\mathbf{x}$ are independent, identically distributed random variables. The common distribution of the $c(\mathbf{x})^2$ is assumed to be absolutely continuous with density ρ supported on the interval $[a, M]$. We say that the high-frequency propagation is *b-bounded* if this density satisfies the bound

$$\rho(E) \leqslant b/E \tag{7}$$

This parameter b will be a measure of the concentration of the distribution, so that small b implies large disorder. Notice that

$$1 = \int_a^M \rho(E)\, dE \leqslant \int_a^M \frac{b}{E}\, dE = b \log \frac{M}{a} \tag{8}$$

Thus, large disorder implies that the ratio of the bounds M/a is large. This may always be accomplished by taking $a > 0$ sufficiently small.

For each site $\mathbf{x}$ in $\mathbb{Z}^\nu$ let $\phi_\mathbf{x}$ be the function that is one at that site and zero elsewhere. This is a unit vector in the Hilbert space $\mathscr{H}$.

Theorem 1. Let $m < \infty$ be a prescribed decay constant. Let $M < \infty$ be an upper bound on the $c(\mathbf{x})^2$. Let $0 < \beta < 1$ be a prescribed fraction. Then there exists a sufficiently small $b > 0$, so that if the high-frequency propagation is b-bounded, then for every E with $4\nu\beta M \leqslant E \leqslant 4\nu M$ there exists a random variable $K < \infty$ so that with probability one for all $\varepsilon > 0$ and all sites $\mathbf{x}$ in $\mathbb{Z}^\nu$,

$$|\langle \phi_\mathbf{x}, (H - E - i\varepsilon)^{-1} \phi_0 \rangle| \leqslant K e^{-m|x|} \tag{9}$$

The theorem says that given a prescribed decay constant $m < \infty$ and a restriction on the range of frequency E, then for sufficiently large disorder the Green's function at fixed frequency E decays exponentially with that decay constant. The rest of this paper is devoted to proving the theorem.

Corollary 2. Assume that the decay constant m is taken with $m > 0$. Then in the situation described in the theorem

$$\langle \phi_0, (H - E)^{-2} \phi_0 \rangle < \infty \tag{10}$$

with probability one.

Proof. By the dominated convergence theorem

$$\langle \phi_0, (H-E)^{-2} \phi_0 \rangle = \lim_{\varepsilon \to 0} \langle \phi_0, [(H-E)^2 + \varepsilon^2]^{-1} \phi_0 \rangle \tag{11}$$

However,

$$\begin{aligned} \langle \phi_0, [(H-E)^2 + \varepsilon^2]^{-1} \phi_0 \rangle &= \|(H-E-i\varepsilon)^{-1} \phi_0\|^2 \\ &= \sum_{\mathbf{x}} |\langle \phi_{\mathbf{x}}, (H-E-i\varepsilon)^{-1} \phi_0 \rangle|^2 \\ &\leqslant K \sum_{\mathbf{x}} e^{-m|x|} < \infty \end{aligned} \tag{12}$$

This completes the proof. ∎

The importance of this corollary is that it has been proved[7] by an extension of the methods of Ref. 10 that this estimate, together with the independence of the $c(\mathbf{x})$ and the absolute continuity of their distributions, implies that the only possible spectral type in this range of frequency is point spectrum. In terms of wave propagation this says that a sufficiently large amount of disorder produces a situation where there are only standing waves at the higher frequencies.

2. THE NONRESONANT REGION

In the following we shall often use the discrete Laplace operator H_A associated with a finite subset A of $\mathbb{Z}^\nu$, with *Dirichlet boundary conditions.* This operator is defined in the Hilbert space $\mathscr{H}_A = l^2(A)$. The only change in the definition is that the terms that couple the region A to its complement are missing.

The first task is to establish exponential decay in a large nonresonant region. Let m_0 be a prescribed decay rate constant. Let $\alpha > 0$ be a parameter defined by

$$\frac{\alpha}{2\nu(2-\alpha)} = e^{-m_0} \frac{1}{2\nu + 1} \tag{13}$$

Thus, small α corresponds to large m_0. Define the -1*th gentle region* S^g_{-1} to be the set of sites $\mathbf{x}$ where only low-frequency propagation is permitted:

$$S^g_{-1} = \{\mathbf{x} \mid c(\mathbf{x})^2 \leqslant \alpha\beta M\} \tag{14}$$

The next result shows that there is exponential decay in this region.

Proposition 3. Assume that $A \subset S^g_{-1}$. Then, for $E \geqslant 4\nu\beta M$

$$|\langle \phi_{\mathbf{x}}, (H_A - E - i\varepsilon)^{-1} \phi_{\mathbf{y}} \rangle| \leqslant C \exp(-m_0 |\mathbf{x}-\mathbf{y}|) \tag{15}$$

where $C = e^{-m_0}/\alpha\beta M = (2\nu+1)/[2\nu(2-\alpha)\beta M]$.

Proof. Write

$$H_A - E - i\varepsilon = D + Q \tag{16}$$

where D is multiplication by $d(\mathbf{x}) = 2\nu c(\mathbf{x})^2 - E - i\varepsilon$ and Q has the matrix element $c(\mathbf{x})\, c(\mathbf{y})$ at nearest neighbor sites $\mathbf{x}$ and $\mathbf{y}$. Then

$$(H_A - E - i\varepsilon)^{-1} = (D+Q)^{-1} = \sum_{n=0}^{\infty} D^{-1}(QD^{-1})^n \tag{17}$$

It follows that $\langle \phi_{\mathbf{x}}, (H_A - E - i\varepsilon)^{-1} \phi_{\mathbf{y}} \rangle$ is the sum over all paths from $\mathbf{x}$ to $\mathbf{y}$ of a product along the sites $\mathbf{z}$ of the path of expressions $c(\mathbf{z})^2/d(\mathbf{z})$, except that the two ends of the path contribute only $c(\mathbf{x})/d(\mathbf{x})$ and $c(\mathbf{y})/d(\mathbf{y})$. We use the estimate

$$\frac{c(\mathbf{z})^2}{|d(\mathbf{z})|} \leqslant \frac{\alpha}{2\nu(2-\alpha)} \tag{18}$$

on the sites interior to the path. The endpoints contribute two factors of $\alpha/[2\nu(2-\alpha)]$ times an extra factor of $1/\alpha\beta M$. From the definition of α we see that we get a factor $\exp(-m_0)/(2\nu+1)$ for each site along the path. Since the number of sites in a path from $\mathbf{x}$ to $\mathbf{y}$ exceeds the distance $|\mathbf{x}-\mathbf{y}|$ by at least one, this gives an exponential decay factor $\exp(-m_0)$ $\exp(-m_0 |\mathbf{x}-\mathbf{y}|)$. The remaining factor is $1/(2\nu+1)$ for each site of the path. This is precisely the path expansion of $\langle \phi_{\mathbf{x}}, (-\Delta+1)^{-1} \phi_{\mathbf{y}} \rangle \leqslant 1$. The remaining constants are the constants in the statement of the theorem. ∎

Now that we have established exponential decay in the -1th gentle set, it would be nice to know that this set is large. The following proposition is a step in this direction.

Proposition 4. For every $\mathbf{x} \in \mathbb{Z}^\nu$

$$P[\mathbf{x} \notin S^g_{-1}] \leqslant b \log(1/\alpha\beta) \tag{19}$$

Proof.

$$P[\mathbf{x} \notin S^g_{-1}] \leqslant P[c(\mathbf{x})^2 > \alpha\beta M] \leqslant \int_{\alpha\beta M}^{M} \frac{b}{E}\, dE = b \log\left(\frac{1}{\alpha\beta}\right) \tag{20}$$

This is the desired estimate. ∎

It follows from this proposition that the proportion of points in the complement of the -1th gentle set S^g_{-1} can be made arbitrarily small by taking b sufficiently small. On the other hand, there will be arbitrarily large finite sets in this complement. Consequently, we will have to examine the behavior of the Green's function on increasingly resonant subsets of the complement.

3. THE RESONANCE BOUND

The crucial step in controlling the Green's function in the more resonant regions is a lemma that shows that it is improbable that the operator norm of the Green's function is large in a moderate-sized region. This is a consequence of the following density of states bound. Such a result was proved for the Schrödinger equation by Wegner.[11] The following proof for the wave equation was suggested by Simon.[12]

Theorem 5. Let $N_A(S)$ be the number of eigenvalues of H_A in the set S. Then

$$\mathscr{E}[N_A(S)] \leqslant b\,|A| \int_S \frac{dE}{E} \tag{21}$$

This result says that the density of states per unit volume and unit frequency is bounded by b/E.

Proof. The proof is based on the following lemma.[7] The lemma is proved using a general theory of rank-one multiplicative perturbations. It says that averaging over the parameter of the perturbation produces a measure that is absolutely continuous with respect to the spectral parameter.

Lemma 6. Fix $\mathbf{x}$ in A. Fix the values of the $c(\mathbf{y})$ for $\mathbf{y} \neq \mathbf{x}$. Let S be a set of positive real numbers, and let $1_S(H_A)$ be the corresponding spectral projection of H_A. Then

$$\int_I \langle \phi_{\mathbf{x}}, 1_S(H_A)\,\phi_{\mathbf{x}}\rangle \frac{dc(\mathbf{x})}{c(\mathbf{x})} \leqslant \frac{1}{2} \int_S \frac{dE}{E} \tag{22}$$

This lemma may be used to bound the conditional expectation of $\langle \phi_{\mathbf{x}}, 1_S(H_A)\,\phi_{\mathbf{x}}\rangle$ given the $c(\mathbf{y})$ with $\mathbf{y} \neq \mathbf{x}$. Since the probability measure is

$$\rho(c(\mathbf{x})^2)\,dc(\mathbf{x})^2 \leqslant \frac{b}{c(\mathbf{x})^2}\,dc(\mathbf{x})^2 = \frac{2b}{c(\mathbf{x})}\,dc(\mathbf{x}) \tag{23}$$

the bound is $b \int_S dE/E$.

It follows that the expectation $\mathscr{E}[\langle\phi_{\mathbf{x}}, 1_S(H_A)\,\phi_{\mathbf{x}}\rangle]$ has the same bound. The theorem follows by summing over $\mathbf{x}$ and noting that the trace of $1_S(H_A)$ is just $N_A(S)$. ∎

Notice that the expectation on the left-hand side of the inequality in the statement of the lemma is an upper bound for the probability $\mathscr{P}[N_A(S) \neq 0]$. Thus, the lemma may be used to show that there is a high probability of a small gap in the spectrum. The following theorem is the resulting *resonance bound.*

Theorem 7. The operator H_A in region A satisfies the norm bound

$$\mathscr{P}[\|(H_A - E)^{-1}\| \geqslant \gamma] \leqslant 4\,\frac{b}{E}\,\frac{|A|}{\gamma} \tag{24}$$

for $\gamma > 2/E$.

Proof. Take $S = [E-\kappa, E+\kappa]$. Then by the theorem

$$\mathscr{P}[N_A([E-\kappa, E+\kappa]) \neq 0] \leqslant b\,|A|\,\frac{2\kappa}{E-\kappa} \tag{25}$$

Take $\kappa = 1/\gamma$. ∎

We want to apply this resonance bound to regions A with large volume $|A|$. In order to have a useful estimate, we must be content with even larger bounds γ.

4. THE RENORMALIZATION CONSTRUCTION

The rest of the argument is to apply the proof that Fröhlich and Spencer[4] gave for the case of the Schrödinger equation to show that the initial decay estimate and the estimate on the probability of resonance imply the final decay estimate. Their proof is intricate, and there are numerous technicalities; there is no reason to repeat it here. However, the ideas are interesting, and so the remainder of the paper contains an outline of some of the essential points.

For one part of the proof it is necessary to put the problem into a finite box (subset of $\mathbb{Z}^\nu$), and impose Dirichlet boundary conditions on the boundary. If the estimates are independent of the box, then they will carry over to all of $\mathbb{Z}^\nu$. From now on, points in $\mathbb{Z}^\nu$ are assumed to be in the box, and subsets of $\mathbb{Z}^\nu$ are assumed to be contained in the box.

The procedure for obtaining the estimate on the decay constant is *renormalization* for distances on increasing scales. One starts with a region in which one already has an estimate on a certain scale. The renormalization is the process of obtaining a new estimate in a larger region for distances exceeding a larger scale, in terms of the estimate in the previous regions.

The regions that are introduced in the process of renormalization are the *gentle regions* S^g_{-1}, S^g_0, S^g_1, S^g_2,... with successively larger *distance scales* d_{-1}, d_0, d_1, d_2,..., and corresponding *resonance bounds* γ_{-1}, γ_0, γ_1, γ_2,... . The distance scales and resonance bounds are prescribed in advance.

The initial gentle region S^g_{-1} was already introduced in Section 2. It is the union of all one-point sets $\{\mathbf{x}\}$ for which $c(\mathbf{x})^2 \leqslant \gamma_{-1}$. Here $d_{-1} = 1$ and $\gamma_{-1} = \alpha\beta M$. The decay constant is already estimated in S^g_{-1}. The other gentle regions will be made out of sets C satisfying certain conditions.

A set $C \subset \mathbb{Z}^\nu$ is said to be *i-small* if its diameter is less than d_i.

A set $C \subset \mathbb{Z}^\nu$ is said to be $i+1$-*surrounded* by another set B if every point in $\mathbb{Z}^\nu$ that is within a distance $2d_{i+1}$ of C is either in B or in C.

If $C \subset \mathbb{Z}^\nu$, then the i-neighborhood $\bar{C}$ is the set of all $\mathbf{x}$ in $\mathbb{Z}^\nu$ within $4d_i$ of C. The set C is said to be *i-nonresonant* if $\|(H_{\bar{C}} - E)^{-1}\| \leqslant \gamma_i$. Notice that this says that the propagation is not resonant in a whole i-neighborhood of C. If this bound is violated, then C is called *i-resonant*.

The renormalized gentle regions S^g_i for $i = 0, 1, 2$,... are defined inductively. Assume that the previous gentle regions S^g_{-1},..., S^g_{i-1} are already defined. The new gentle region S^g_i is defined as the maximal union of nonempty disjoint sets C that are disjoint from the previous gentle regions and satisfy:

(a) C is i-small.

(b) C is $i+1$-surrounded by the union of the previous gentle regions S^g_{-1},..., S^g_{i-1}.

(c) C is i-nonresonant.

This process terminates: the finite box is contained in the union of finitely many of the gentle regions S^g_i. In fact, eventually, for some i, the diameter of the box will be exceeded by d_i, and the resonance of the box will be less than γ_i. If there are points that are not in the union of the previous gentle regions, then they form a component of the final gentle region S^g_i. (Of course, the estimates cannot depend on when the process terminates, since then they would depend on the box.)

It is possible to estimate inductively the decay of the Green's function in the gentle regions. For this reason, a point in or subset of $\mathbb{Z}^\nu$ is said to be *i-renormalized* if it belongs to or is contained in the union of the gentle regions S^g_{-1},..., S^g_{i-1}.

5. THE RENORMALIZATION

The conditions (a)–(c) of the renormalization construction are used to control the renormalization that passes from scale k to scale $k+1$. There is an inductive assumption on the size of the decay constant in the union of the sets $S^g_{-1},..., S^g_{k-1}$ for distance scales exceeding d_k.

The inductive assumption is that at stage k the Green's function is estimated for distances exceeding a multiple of d_k in every k-renormalized region A that satisfies certain technical conditions. The decay estimate on the Green's function of H_A is that

$$|\langle \phi_{\mathbf{x}}, (H_A - E - i\varepsilon)^{-1} \phi_{\mathbf{y}} \rangle| \leqslant \exp(-m_k |\mathbf{x}-\mathbf{y}|) \tag{26}$$

for $|\mathbf{x}-\mathbf{y}| \geqslant d_k/5$.

This assumption is certainly true for $k=0$, since then such a region A is contained in the nonresonant region S^g_{-1}.

The inductive step is to pass to a similar estimate for $k+1$-renormalized regions. Let A be such a region. By conditions (a) and (b) the components C in S^g_k are small on scale d_k and surrounded by already renormalized regions on scale d_{k+1}. Therefore the decay of the Green's function in the union of the sets $S^g_{-1},..., S^g_{k-1}$ on scales exceeding d_{k+1} compensates for the resonance in C, which is bounded by condition (c).

The compensation is estimated by a perturbation argument. The magnitude of the coupling between adjacent sites $\mathbf{x}$ and $\mathbf{y}$ is bounded by $|c(\mathbf{x})\, c(\mathbf{y})| \leqslant M$. Thus, introducing a component C of S^g_k has an effect that is determined by the geometry of the component, which is of size d_k, and by the degree of resonance, which is γ_k. The geometric factor has the effect of replacing the $m_k |\mathbf{x}-\mathbf{y}|$ in the exponent by $m_k |\mathbf{x}-\mathbf{y}| - Cm_k d_k$ for some constant C. The correction factor $m_k d_k$ satisfies the estimate

$$m_k d_k \leqslant m_k(d_k/d_{k+1})\, d_{k+1} \leqslant 5m_0(d_k/d_{k+1})\, |\mathbf{x}-\mathbf{y}| \tag{27}$$

for $|\mathbf{x}-\mathbf{y}| \geqslant d_{k+1}/5$. The new estimate on the decay constant is thus

$$m_{k+1} = m_k - C' d_k/d_{k+1} \tag{28}$$

for another constant C'.

This means that for the sum of all the corrections to the estimate of the decay constant to be finite, the condition on the distances is given by

$$\sum_{k=1}^{\infty} (d_k/d_{k+1}) < \infty \tag{29}$$

In fact, the sum must be small enough so that all the m_k stay strictly above zero. This may be accomplished by taking the d_k to increase very rapidly, according to the rule

$$d_k = d_0^{\rho^k} \tag{30}$$

where the distance growth parameter $\rho > 1$, and by taking a sufficiently large d_0.

The resonances must be small enough so that the resulting correction is dominated by the geometric correction. This is ensured by taking

$$\gamma_k = \exp d_k^\tau \tag{31}$$

with the resonance growth parameter $\tau < 1$.

Since each $m_k \geqslant m > 0$, the final result is that at each scale k, the Green's function is controlled on scales exceeding d_k in every k-renormalized region A that satisfies certain technical conditions. Namely, the Green's function of H_A satisfies a decay estimate

$$|\langle \phi_{\mathbf{x}}, (H_A - E - i\varepsilon)^{-1} \phi_{\mathbf{y}} \rangle| \leqslant \exp(-m\,|\mathbf{x}-\mathbf{y}|) \tag{32}$$

for $|\mathbf{x}-\mathbf{y}| \geqslant d_k/5$.

6. RESONANT SUBSETS

The other half of the argument is to show that this renormalization process converges. In other words, the problem is to show that a fixed region is likely to be renormalized after a finite number of steps, that is, to be contained in the union of a finite number of renormalized regions S_k^g.

In order to accomplish this, it is necessary to show that the S_k^g have low density for large k. The reason for this low density is that the construction ensures that certain subsets of the S_k^g are resonant, and resonance is improbable.

A set C is said to be $k+1$-*isolated* from another set D if C is a distance at least $2d_{k+1}$ from D.

Lemma 8. Let R be a nonempty subset of $\mathbb{Z}^\nu$. Let C be a component of the set S_k^g. Assume that $R \subset C$. Consider $j < k$. Assume that R is j-small. Assume also that R is $j+1$-isolated from $C \backslash R$. Then R is j-resonant.

Sublemma. R is $j+1$-surrounded by the union of previous gentle sets $S_{-1}^g, \ldots, S_{j-1}^g$.

Proof of Sublemma. Since $R \subset C \subset S_k^g$, it follows that R is $k+1$-surrounded by the union of the gentle sets $S_{-1}^g, \ldots, S_{k-1}^g$, together with $C \backslash R$. In particular, it is $j+1$-surrounded by this set.

By assumption, $C \backslash R$ is $j+1$-isolated from R. Hence, R is $j+1$-surrounded by the union of the gentle sets $S^g_{-1},..., S^g_{k-1}$.

Now consider a component of some S^g_r with $j \leqslant r \leqslant k-1$. Such a component is $r+1$-surrounded by the union of the gentle sets $S^g_{-1},..., S^g_{r-1}$ and so is $r+1$-isolated from S^g_k. In particular, it is $j+1$-isolated from R. Hence R is $j+1$-surrounded by the union of the gentle sets $S^g_{-1},..., S^g_{j-1}$. ∎

Proof of Lemma. If R were j-nonresonant, then by the sublemma it would satisfy all the conditions to be a component of S^g_j. Since S^g_j is a maximal union of such subsets, R would have already been included as a component of S^g_j. This is a contradiction. ∎

The importance of this lemma is that it gives a bound on the probability of such a component C of S^g_j with $R \subset C$. In fact, since R is j-small and j-resonant, we have the resonance bound $(4b/E)(d^\nu_j)/\gamma_j$.

Corollary 9. Let R be a nonempty subset of $\mathbb{Z}^\nu$. If R is a component of S^g_k that is $k-1$-small, then R is $k-1$-resonant.

7. CONVERGENCE OF THE RENORMALIZATION

The goal is to show that the renormalization process converges in reasonable-sized regions with high probability. In other words, the goal is to show that for large k the probability that a reasonable-sized region is not k-renormalized is small. This is accomplished by a series of propositions. The starting point is to show that the probability that a fixed region D is a component of S^g_k is small.

The lemma on resonant subsets shows that for large k a fixed set R is unlikely to be a small, isolated subset of a component of S^g_k, since then it would be resonant. It follows that a large set D is very unlikely to be a component of S^g_k. The reason is that such a set would have many small, isolated subsets, and it would be very unlikely that they would all be resonant.

Proposition 10. Fix a nonempty subset D of $\mathbb{Z}^\nu$. For every $q < \infty$ there is a constant $C < \infty$ and a constant $b > 0$ such that when the propagation satisfies the b-bound, then

$$P[D \text{ is a component of } S^g_k] \leqslant C/d^q_k \tag{33}$$

Proof Outline. There are two cases to consider: either D is $k-1$-small, or it is not.

If D is not $k-1$-small, then the statistical mechanical analysis of Fröhlich and Spencer[4] shows that D has many small, isolated subsets R

on various smaller scales, down to scale -1. For scales $j \geqslant 0$ these subsets R are each j-small and $j+1$-isolated from $D \backslash R$ for some $j<k$. It follows from the lemma on resonant subsets that each such R is j-resonant for the appropriate j.

Since on each scale j the R are $j+1$-isolated, their resonances are independent events. Hence the probability that they are all resonant is the product of the probabilities. These probabilities are individually small. In fact, they are given by a small constant times $d_j^\nu \gamma_j^{-1}$ for $j \geqslant 0$ (by the resonance bound), and by a small constant for $j=-1$.

The different scales j are not independent, but the probability of the event $\mathscr{E}$ of resonances in the isolated subsets on all scales may be estimated in terms of the probabilities of the events $\mathscr{E}_j$ of resonances on the scales j. Let r_j be a sequence with $\sum r_j = 1$. Since $\mathscr{E}$ is the intersection of the $\mathscr{E}_j$, we have the estimate

$$P[\mathscr{E}] \leqslant \prod_{j=-1}^{\infty} P[\mathscr{E}_j]^{r_j} \tag{34}$$

The factors r_j approach zero. However, if the probabilities decrease very rapidly with j, the probability that D is a component of S_k^g will still be bounded by a product of individual small numbers.

A sufficiently rapid decrease for the above argument to work is that the resonance growth parameter $\tau > 0$. The small factors on scale $j \geqslant 0$ are then $d_j^\nu \gamma_j^{-1} = d_j^\nu \exp(-d_j^\tau)$.

The estimate on the number of small, isolated subsets is based on the assumption that the distance growth parameter $\rho < 2$, so the spacing between distance scales is not too large. The result is that if D is not $k-1$-small, the number of these subsets exceeds a constant C times $\log d_k$. Thus, if e^{-k} is the small factor for such a subset, then the probability is bounded by $\exp(-Ck \log d_k) = d_k^{-Ck}$.

The other case is when D is $k-1$-small. By the corollary on resonant subsets, D is $k-1$-resonant. This has a very small probability, by the resonance bound. ∎

Proposition 11. Fix $\mathbf{x} \in \mathbb{Z}^\nu$. For every $q < \infty$ there is a constant $C < \infty$ and a constant $b > 0$ such that when the propagation satisfies the b-bound, then

$$P[\mathbf{x} \in S_k^g] \leqslant C/d_k^q \tag{35}$$

Proof Outline. The probability to estimate is

$$\sum_{\mathbf{x} \in D} P[D \text{ is a component of } S_k^g] \tag{36}$$

where D ranges over subsets of $\mathbb{Z}^\nu$. The new problem is to show that the number of subsets D with many small, isolated subsets is dominated by the small probability that such subsets exist. In fact, Fröhlich and Spencer show that this summing over subsets merely contributes a fixed multiplicative factor for each such subset. ∎

Proposition 12. Fix $\mathbf{x} \in \mathbb{Z}^\nu$. For every $p < \infty$ there is a constant $C < \infty$ and a constant $b > 0$ such that when the propagation satisfies the b-bound, then

$$P[\mathbf{x} \text{ is not } k\text{-renormalized}] \leqslant C/d_k^p \tag{37}$$

Proof. This follows from the fact that $\mathbf{x}$ is not k-renormalized if and only if $\mathbf{x}$ is in the union of the S_j^g for $j \geqslant k$. Therefore, the probability of this event is bounded by

$$\sum_{j=k}^{\infty} P[\mathbf{x} \in S_j^g] \leqslant \sum_{j=k}^{\infty} C/d_j^q \leqslant C/d_k^p \tag{38}$$

if q is taken large enough. ∎

Proposition 13. For every p, $r < \infty$ there exists a constant $C < \infty$ and a constant $b > 0$ such that when the propagation satisfies the b-bound, then for every region A with diam $A \leqslant d_k^r$,

$$P[A \text{ is not } k\text{-renormalized}] \leqslant C/d_k^p \tag{39}$$

Proof. The probability to be estimated is bounded by

$$\sum_{\mathbf{x} \in A} P[\mathbf{x} \text{ is not } k\text{-renormalized}] \leqslant C d_k^{\nu r}/d_k^q \tag{40}$$

Take q such that $q - \nu r = p$. ∎

Let $m < \infty$ be the prescribed decay constant. We know from the renormalization argument that in k-renormalized regions A the Green's function satisfies the decay bound for $|\mathbf{x} - \mathbf{y}| \geqslant d_k/5$. Thus, the probability of a violation of the bound (a large matrix element) in a region A is bounded by the probability that the region is not k-renormalized.

Proposition 14. Let $m < \infty$ be a prescribed decay constant. For every $p < \infty$ there exists a constant $C < \infty$ and a constant $b > 0$ such that when the propagation satisfies the b-bound and E is in the appropriate interval, then for every $N < \infty$,

$$P[\exists \mathbf{x}, \varepsilon\colon \ |\langle \phi_{\mathbf{x}}, (H - E - i\varepsilon)^{-1} \phi_0 \rangle| > e^{m(N - |\mathbf{x}|)}] \leqslant C/N^p \tag{41}$$

Proof Outline. There are two cases.

If $|\mathbf{x}| \geqslant N/2$, then use a perturbation argument on all scales larger than $\mathbf{x}$. The probability of a violation on some scale is bounded by the sum of the probabilities of violations on each scale. The previous proposition then shows that a matrix element exceeding $e^{-m|\mathbf{x}|}$ is unlikely.

If $|\mathbf{x}| \leqslant N/2$, then use the same proof on scales larger than $N/2$. On scale $N/2$ the resonance bound is used to show that a norm exceeding $e^{mN/2}$ is unlikely, and hence a matrix element $e^{m(N-|\mathbf{x}|)}$ is unlikely. ∎

Theorem 1 follows from this proposition, by the following argument (Borel–Cantelli lemma). The probability that the bound is violated for infinitely many N is bounded, for each N_0, by the probability that the bound is violated for some $N \geqslant N_0$, which in turn is bounded by the sum $\sum_{N=N_0}^{\infty} C/N^p$. This sum is arbitrarily small when N_0 is large enough. Therefore the probability that the bound is violated for infinitely many N must be zero.

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